A Generalized Mixed Zero-sum Stochastic Differential Game and Double Barrier Reflected BSDEs with Quadratic Growth Coefficient

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Abstract

This article is dedicated to the study of mixed zero-sum two-player stochastic differential games in the situation when the player's cost functionals are modeled by doubly controlled reflected backward stochastic equations with two barriers whose coefficients have quadratic growth in Z. This is a generalization of the risk-sensitive payoffs. We show that the lower and the upper value function associated with this stochastic differential game with reflection are deterministic and they are also the unique viscosity solutions for two Isaacs equations with obstacles.

Keywords and phrases: Backward stochastic differential equations with two reflecting barriers and quadratic growth ; HJB PDEs with two obstacles ; Viscosity solutions ; Mixed zero-sum stochastic differential games.

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1 Introduction

An American option is a contract which gives the right to its holder to exercise, i.e. to ask for the wealth, when he decides before the maturity of the option. The liabilities of the seller of the option is to provide this wealth and nothing else. In recent last years there have been other options, which have attracted a lot of research activity, which look like to American options but they give also the right to the seller of the option to recall it if he accepts to pay a money penalty, namely American game options. Those types of options, introduced by Y.Kifer in [24] who terms them as Israeli options, have been well documented in several papers of which we can quote [23], [3], [15], [26]. The main reason for the introduction of those options is that brokers face high level risky environment/market and therefore introduce clauses which allow them to withdraw from their liabilities in case when for example a stock or a commodity, such as crude oil, copper, steel,..., becomes more and more expensive. In order to tackle American options we mainly use Snell envelope of processes while in order to do so for the American game options we use value functions of zero-sum Dynkin games. In the standard

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Black and Scholes model, the value of the American game option is given by the value of the zero-sum Dynkin game under the risk neutral probability. Additionally a hedging strategy for the seller of the option exists (see e.g. [24, 15, 23] for more details).

Now let us introduce a specific stochastic Ramsey model in a growth model in finance (see e.g. [1] for more details). Assume we have a capital from which is withdrawn a consumption and whose dynamics is given by:

$$dX_t = X_t(r - c_t)dt + X_t\sigma dW_t,\ t \le T \text{ and } X_0 = x > 0.$$

Here r is the spot mean-return of the capital and c is a proportion of the capital which is consumed. Therefore usually the main objective is to find an optimal consumption process with respect to an index which indicates the satisfaction of the capital holder. This index depends of course on $(c_t)_{t\le T}$ but also on X and many other parameters such as risk sensitiveness, utility and so on.

The problem we consider in this paper is in a way a generalization of the combination of the two previous ones. Additionally we assume that the criterion is of risk-sensitive type. Actually assume we have a system on which intervene two agents a_1 and a_2 and whose dynamics is given by:

$$(1)\qquad \begin{cases} dX_s^{\alpha,\beta} = b(s, X_s^{\alpha,\beta}, \alpha_s, \beta_s)ds + \sigma(s, X_s^{\alpha,\beta}, \alpha_s, \beta_s)dW_s,\ \ s \in [0, T]\,; \\ X_0^{\alpha,\beta} = x. \end{cases}$$

The stochastic processes $(\alpha_t)_{t\le T}$ and $(\beta_t)_{t\le T}$ are the intervention functions of a_1 and a_2 respectively. The agents are also allowed to stop controlling at stopping times σ for a_1 and τ for a_2. When one of them decides to stop controlling first, the control of the system is stopped. The interests of the agents are antagonistic and there is a payoff whose expression is given by:

$$(2)\qquad \begin{aligned} \Gamma(\alpha, \sigma; \beta, \tau) := \mathbb{E}\Bigg[exp\{\int_0^{T\wedge\tau\wedge\sigma} & \varphi(s, X_s^{\alpha,\beta}, \alpha_s, \beta_s)ds + h(\sigma, X_\sigma^{\alpha,\beta})1_{[\sigma\le\tau<T]} \\ & + h'(\tau, X_\tau^{\alpha,\beta})1_{[\tau<\sigma]} + g(X_T^{\alpha,\beta})1_{[\sigma=\tau=T]}\}\Bigg]. \end{aligned}$$

This payoff $\Gamma(\alpha, \sigma; \beta, \tau)$ is a reward for a_1 and a cost for a_2, therefore the first agent aims at maximizing it while the second one, his objective is to minimize the same quantity. The role of the exponential utility function is to capture the sensitiveness with respect to risk of the agents. To day there have been several papers which deal with risk-sensitive control/games (see e.g. [12, 4, 5, 28, 11]). The risk neutral game corresponds to the case when the payoff equals to the expectation of the quantity inside exponential in (2). So one of the objectives of this paper is to study the upper and lower values of this mixed zero-sum stochastic differential game which are defined by:

$$(3)\qquad \sup_\alpha \inf_\beta \sup_\sigma \inf_\tau \Gamma(\alpha, \sigma; \beta, \tau) \quad \text{and} \quad \inf_\beta \sup_\alpha \inf_\tau \sup_\sigma \Gamma(\alpha, \sigma; \beta, \tau).$$

When the data of this problem do not depend on the controls α and β and the criterion is of risk-neutral type, the problem reduces to the well known zero-sum Dynkin game which is involved when dealing with American game options. On the other hand, in order to obtain the Ramsey model it is enough to take $\tau = \sigma = T$ and assume that the data do not depend on β.

Generally speaking, in this paper we are going to consider a more general setting of payoffs, namely payoffs defined by solutions of BSDEs with two reflecting barriers and continuous coefficients whose growth are quadratic with respect to the component z for which the payoff (2) is somehow a particular case.

The BSDEs with two reflecting barriers have been introduced by Cvitanic-Karatzas in [10]. They have generalized a previous work by El-Karoui et al. [13] on BSDEs with only one reflecting barrier. Since then BSDEs with two reflecting barriers have attracted a lot of research activity especially in connection with zero-sum Dynkin games and American game options (see [2, 16, 17, 15]). In [17], the

authors have shown that if the barriers are completely separated and the coefficient of the BSDE is continuous with quadratic growth, then a minimal and a maximal solution exist for the BSDE. This is the framework which we adopt along with this paper.

For decades there have been a lot of research activity on stochastic differential games (see e.g. [6, 7, 8, 14, 12, 18, 19, 20, 21, 22],... and the references therein). The lower and upper values of a zero-sum stochastic differential game have been already investigated by Fleming-Souganidis in [14]. They proved that they are unique solutions in viscosity sense of their associated Hamilton-Jacobi-Bellman-Isaacs equations. The upper and lower values of a zero-sum mixed differential game have been studied in [20, 17, 22] in connection with reflected BSDEs. For this latter type of games, another framework is the one considered in [8]. The authors studied the mixed zero-sum differential game when the dynamics of the controlled system is solution of (1) and the payoffs are given by a solution of a controlled BSDE with two reflecting barriers whose coefficient is Lipschitz in (y, z). They have shown that the values of the game are unique viscosity solutions of their related Hamilton-Jacobi-Bellman equations. In this work we mainly focus on the lower and upper values of the zero-sum stochastic differential game when the payoffs are given by a solution of a two reflecting barrier BSDEs with continuous coefficients and quadratic growth condition *w.r.t.* z. A particular case of those BSDEs is connected with the risk-sensitive payoff defined in (2). The dynamics of the controlled system is given by (1). We show that those values are unique solutions of their associated Hamilton-Jacobi-Bellman equations.

This paper is organized as follows:

In Section 2, we recall the main results related to BSDEs with two reflecting barriers, while in Section 3 and Section 4, we give the main result of the paper. We show that the lower and upper values are unique viscosity solutions of their respective HJB equations. We begin to show the much more involved issue of uniqueness and later the one of existence. In Section 5, we make the connection between the payoffs Γ given in (2) with the payoffs defined through solutions of a specific BSDEs with two reflecting barriers considered along with this article. $\Box$

2 Preliminaries. Notations. Hypothesis

The purpose of this section is to introduce some basic notations and results concerning RBSDEs with two barriers, which will be needed throughout this paper. In all that follows we shall consider a finite horizon $T > 0$ and a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a standard d-dimensional Brownian motion $W = (W_t)_{t \leq T}$ whose natural filtration is denoted $\mathbb{F} =\{\mathcal{F}_t,\ 0 \leq t \leq T\}$. More precisely, $\mathbb{F}$ is the filtration generated by the process W and augmented by $\mathcal{N}_{\mathbb{P}}$, the set of all $\mathbb{P}$-null sets, *i.e.* $\mathcal{F}_t = \sigma\{W_s,\ s \leq t\} \vee \mathcal{N}_{\mathbb{P}}$.

Let us consider:

(i) $\mathcal{P}$, the σ-algebra of $\mathcal{F}_t$-progressively measurable sets on $[0, T] \times \Omega$;

(ii) $\mathcal{L}^{2,k}$, the set of $\mathcal{P}$-measurable and $\mathbb{R}^k$-valued processes $z = (z_t)_{t \leq T}$ such that $\int_0^T |z_t|^2 dt < \infty$, $\mathbb{P}$-a.s.; $\mathcal{H}^{2,k}$ is the subspace of $\mathcal{L}^{2,k}$, such that $\mathbb{E}[\int_0^T |z_t|^2 dt] < \infty$;

(iii) $\mathcal{S}^2$, the set of $\mathcal{P}$-measurable and continuous processes $Y = (Y_t)_{t \leq T}$ such that $\mathbb{E}[\sup_{t \leq T} |Y_t|^2] < \infty$;

(iv) $\mathcal{M}$, the set of continuous $\mathcal{P}$-measurable nondecreasing processes $(K_t)_{t \leq T}$ such that $K_0 = 0$ and $K_T < \infty$, $\mathbb{P}$-a.s.

Let us now recall the existence result for the solutions of RBSDEs with two barriers with quadratic growth coefficient. For that, let us take four objects which define the equation:

- a continuous function $F : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ (also known as the *coefficient or generator* of the equation) which grows subquadratically with respect to z, i.e., there exists a constant $C > 0$ such that, $\mathbb{P}$-a.s.,

$$|F(t,\omega,y,z)| \leq C(1+|z|^2), \ \forall\, (t,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d; \tag{4}$$

- a terminal value ξ, which is a $\mathcal{F}_T$-measurable random variable;

- two processes $U = (U_t)_{t\leq T}$ and $L = (L_t)_{t\leq T}$ from $\mathcal{S}^2$, satisfying $L_t < U_t$, for all $t \leq T$, and $L_T \leq \xi \leq U_T$.

Definition 1 *A solution for the RBSDE associated with (F,ξ,L,U) is a quadruple of $\mathcal{P}$-measurable processes $\left(Y_t, Z_t, K_t^+, K_t^-\right)_{t\leq T}$ from $\mathcal{S}^2 \times \mathcal{L}^{2,d} \times \mathcal{M} \times \mathcal{M}$ such that, $\mathbb{P}$-a.s.*

$$Y_t = \xi + \int_t^T F\left(s, Y_s, Z_s\right) ds + \left(K_T^+ - K_t^+\right) - \left(K_T^- - K_t^-\right) - \int_t^T Z_s dW_s, \ \forall t \leq T$$

and

$$L_t \leq Y_t \leq U_t, \ \forall t \leq T, \quad \int_0^T \left(Y_s - L_s\right) dK_s^+ = \int_0^T \left(U_s - Y_s\right) dK_s^- = 0.$$

We will also assume that U, L and ξ are bounded, i.e.,

$$\operatorname{esssup}[|\xi| + \sup_{t\leq T}\{|U_t| + |L_t|\}] < +\infty. \tag{5}$$

We have the following results (see, *i.e*, [17], Theorem 3.2 and Remark 3.3).

Theorem 2 *Under the assumptions (4) and (5), there exists a $\mathcal{P}$-measurable process (Y, Z, K^+, K^-) solution for the RBSDE associated with (F,ξ,L,U). Moreover, the solution is maximal, i.e., if (Y', Z', K'^+, K'^-) is another solution of the above equation, then $\mathbb{P}$-a.s., for all $t \leq T$, we have $Y_t' \leq Y_t$.*

The main idea for proving the existence of a solution for RBSDE associated with (F,ξ,L,U) is to find a solution for a RBSDE associated with data obtained by an exponential transform of (F,ξ,L,U). Then, by a limiting procedure applied to a suitably constructed monotone sequence of bounded, continuous generators which approximate f, it is shown that the initial equation has a solution. We will use a similar technique for the proof of our existence result for the Isaacs equation associated with stochastic games. Another useful result is the following comparison principle.

Proposition 3 (comparison) *Let F and F' be two generators satisfying (4) and (5) such that, $\mathbb{P}$-a.s., $F\left(t,\omega,y,z\right) \leq F'\left(t,\omega,y,z\right)$ for any t, y, z, and consider $\left(Y_t, Z_t, K_t^+, K_t^-\right)_{t\leq T}$, (resp. $\left(Y_t', Z_t', K_t'^+, K_t'^-\right)_{t\leq T}$) the maximal solution of the RBSDE associated with (F,ξ,L,U) (resp. (F',ξ,L,U)). Then, $\mathbb{P}$-a.s. $Y_t \leq Y_t'$, for all $t \leq T$.*

3 Main results

We introduce now the framework for the study of stochastic differential games with reflection for two players and we give the results which assert the existence and uniqueness of viscosity solutions for the associated Isaacs equations with obstacles. The proofs of these results will be detailed in the next section.

3.1 The setting of the problem

Let A and B be two compact metric spaces.

Definition 4 *An admissible control process $\alpha = (\alpha_s)_{s\in[t,T]}$ (resp., $\beta = (\beta_s)_{s\in[t,T]}$) for Player I (resp., Player II) on $[t,T]$ $(t < T)$ is an $\mathbb{F}$-progressively measurable process taking values in A (resp., B). The set of all admissible controls on $[t,T]$ for the two players will be denoted by $\mathcal{A}_t$, respectively $\mathcal{B}_t$.*

Now, for $t < T$, $\alpha(\cdot) \in \mathcal{A}_t$ and $\beta(\cdot) \in \mathcal{B}_t$, let us consider the following SDE:

$$
\left\{\begin{array}{l} dX_s^{t,x;\alpha,\beta} = b(s, X_s^{t,x;\alpha,\beta}, \alpha_s, \beta_s)ds + \sigma(s, X_s^{t,x;\alpha,\beta}, \alpha_s, \beta_s)dW_s,\ s \in [t,T]; \\ X_s^{t,x;\alpha,\beta} = x,\ s \le t, \end{array}\right. \tag{6}
$$

where the coefficients

$$b : [0,T] \times \mathbb{R}^n \times A \times B \longrightarrow \mathbb{R}^n \text{ and } \sigma : [0,T] \times \mathbb{R}^n \times A \times B \longrightarrow \mathbb{R}^{n\times d}$$

satisfy the following condition:

$$
\left\{\begin{array}{ll} (i) & \text{for every } x \in \mathbb{R}^n,\ b(\cdot,x,\cdot,\cdot) \text{ and } \sigma(\cdot,x,\cdot,\cdot) \text{ are continuous; moreover,} \\ (ii) & \text{there exists } C_L > 0 \text{ such that, for all } t \in [0,T],\ x,x' \in \mathbb{R}^n,\ \alpha \in A,\ \beta \in B, \\ & |b(t,x,\alpha,\beta) - b(t,x',\alpha,\beta)| + |\sigma(t,x,\alpha,\beta) - \sigma(t,x',\alpha,\beta)| \le C_L|x - x^{'}|. \end{array}\right. \tag{H1}
$$

It is clear (for example, see [27]) that, under the assumptions (H1), for every $(\alpha(\cdot), \beta(\cdot)) \in \mathcal{A}_t \times \mathcal{B}_t$, the SDE (6) has a unique solution. Moreover, for every $p \ge 2$, there exists $C_p > 0$ such that, for all $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, $(\alpha(\cdot), \beta(\cdot)) \in \mathcal{A}_t \times \mathcal{B}_t$, we have, a.s.:

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]} |X_s^{t,x;\alpha,\beta} - X_s^{t,x';\alpha,\beta}|^p \middle| \mathcal{F}_t\right] \le C_p|x - x'|^p; \\
&\mathbb{E}\left[\sup_{s\in[t,T]} |X_s^{t,x;\alpha,\beta}|^p \middle| \mathcal{F}_t\right] \le C_p(1 + |x|^p);
\end{aligned}
$$

one can see e.g. [7, 8] for more details. The constant C_p depends only on the Lipschitz and the linear growth constants of b and σ.

Let us now consider the functions

$$g : \mathbb{R}^n \longrightarrow \mathbb{R},\ h, h' : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R},\ F : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times A \times B \longrightarrow \mathbb{R}$$

that satisfy the following conditions:

$$
\left\{\begin{array}{l} (i) \quad g \text{ is continuous and bounded; } h \text{ and } h^{'} \text{ are also continuous and bounded} \\ \text{and, for any } (t,x) \in [0,T] \times \mathbb{R}^n,\ h(t,x) < h'(t,x). \text{ Moreover, we assume that} \\ \qquad\qquad h(T,x) \le g(x) \le h'(T,x),\ \forall x \in \mathbb{R}^n; \\ (ii) \quad F \text{ is continuous and has quadratic growth in } z,\ i.e. \\ |F(t,x,y,z,\alpha,\beta)| \le C(1 + |z|^2),\ \forall (t,x,y,z,\alpha,\beta) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times A \times B. \end{array}\right. \tag{H2}
$$

Under the above hypothesis, for any $(t,x) \in [0,T] \times \mathbb{R}^n$ and $(\alpha(\cdot), \beta(\cdot)) \in \mathcal{A}_t \times \mathcal{B}_t$, there exists a maximal solution $\left(Y^{t,x;\alpha,\beta}, Z^{t,x;\alpha,\beta}, K^{+,t,x;\alpha,\beta}, K^{-,t,x;\alpha,\beta}\right)$ of the RBSDE associated with

$$
\Big(F\left(\cdot, X^{t,x;\alpha,\beta}, \cdot, \cdot, \alpha(\cdot), \beta(\cdot)\right), g(X_T^{t,x;\alpha,\beta}), h\left(\cdot, X^{t,x;\alpha,\beta}\right), h'\left(\cdot, X^{t,x;\alpha,\beta}\right)\Big), \tag{7}
$$

where $X^{t,x;\alpha,\beta}$ is the solution of equation (6).

Definition 5 *A nonanticipative strategy for Player I on $[t,T]$ is an application $S_1 : \mathcal{B}_t \longrightarrow \mathcal{A}_t$ such that, for any $\mathbb{F}$-stopping time $\tau : \Omega \longrightarrow [t,T]$ and any $\beta_1(\cdot), \beta_2(\cdot) \in \mathcal{B}_t$ satisfying $\beta_1 = \beta_2$ on $[t,\tau]$, $\mathbb{P} \otimes dt$ a.e., we have that $S_1(\beta_1) = S_1(\beta_2)$ on $[t,\tau]$, $\mathbb{P} \otimes dt$ a.e. A nonanticipative strategy on $[t,T]$ for the second player is a function $S_2 : \mathcal{A}_t \longrightarrow \mathcal{B}_t$ defined in the same manner. We will denote the sets of nonanticipative strategies for the two players by $\mathbb{A}_t$, respectively $\mathbb{B}_t$.*

For any given control processes $\alpha(\cdot) \in \mathcal{A}_t$ and $\beta(\cdot) \in \mathcal{B}_t$, we consider the associated cost functional

$$J(t,x;\alpha,\beta) := Y_t^{t,x;\alpha,\beta}, \quad (t,x) \in [0,T] \times \mathbb{R}^n \tag{8}$$

and we define the *lower value function* of the stochastic differential game with reflection

$$\mathcal{W}(t,x) := \operatorname*{essinf}_{S_2 \in \mathbb{B}_t} \operatorname*{esssup}_{\alpha \in \mathcal{A}_t} J(t,x;\alpha,S_2(\alpha)) \tag{9}$$

and the *upper value function*

$$\mathcal{V}(t,x) := \operatorname*{esssup}_{S_1 \in \mathbb{A}_t} \operatorname*{essinf}_{\beta \in \mathcal{B}_t} J(t,x;S_1(\beta),\beta). \tag{10}$$

Remark 6 *The essential infimum and the essential supremum exist and should be understood with respect to indexed families of random variables (see the appendix of [27], pp. 323–325).*

Let us now introduce the following two Isaacs equations with obstacles

$$\begin{cases} \min\left\{u(t,x) - h(t,x), \max\left\{-\dfrac{\partial u}{\partial t}(t,x) - H^-\left(t,x,u,Du,D^2u\right), u(t,x) - h'(t,x)\right\}\right\} = 0; \\ u(T,x) = g(x), \end{cases} \tag{11}$$

$$\begin{cases} \min\left\{v(t,x) - h(t,x), \max\left\{-\dfrac{\partial v}{\partial t}(t,x) - H^+\left(t,x,v,Dv,D^2v\right), v(t,x) - h'(t,x)\right\}\right\} = 0; \\ v(T,x) = g(x), \end{cases} \tag{12}$$

associated with the Hamiltonians

$$\begin{aligned} &H^-(t,x,u,q,X) \\ &\qquad := \sup_{\alpha \in A} \inf_{\beta \in B} \left\{\frac{1}{2} Tr\left(\sigma\sigma^T(t,x,\alpha,\beta) X\right) + \langle b(t,x,\alpha,\beta), q\rangle + F(t,x,u,q\sigma(t,x,\alpha,\beta),\alpha,\beta)\right\} \end{aligned}$$

and

$$\begin{aligned} &H^+(t,x,u,q,X) \\ &\qquad := \inf_{\beta \in B} \sup_{\alpha \in A} \left\{\frac{1}{2} Tr\left(\sigma\sigma^T(t,x,\alpha,\beta) X\right) + \langle b(t,x,\alpha,\beta), q\rangle + F(t,x,u,q\sigma(t,x,\alpha,\beta),\alpha,\beta)\right\}, \end{aligned}$$

for all $(t,x,u,q,X) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$ ($\mathbb{S}_n$ denotes the set of symmetric $n \times n$ matrices).

The purpose of this article is to show that, under suitable hypothesis, the functions $\mathcal{W}$ and $\mathcal{V}$ are the unique viscosity solutions of equations (11), respectively (12). This section is dedicated to a short survey of the main results; the detailed proofs follow in the next section.

Definition 7 *1. An upper semicontinuous function $u : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a viscosity subsolution of equation (11) if $u(T,x) \le g(x)$, for every $x \in \mathbb{R}^n$, and whenever $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n)$ and $(t,x) \in [0,T) \times \mathbb{R}^n$ is a maximum point for $u - \varphi$, we have*

$$\min\left\{u(t,x) - h(t,x), \max\left\{-\frac{\partial \varphi}{\partial t}(t,x) - H^-\left(t,x,u,D\varphi,D^2\varphi\right), u(t,x) - h'(t,x)\right\}\right\} \le 0;$$

2. *A lower semicontinuous function $u : [0,T) \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a viscosity supersolution of equation (11) if $u(T,x) \geq g(x)$, for every $x \in \mathbb{R}^n$, and whenever $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n)$ and $(t,x) \in [0,T) \times \mathbb{R}^n$ is a minimum point for $u - \varphi$, we have*
$$\min\left\{u(t,x) - h(t,x), \max\left\{-\frac{\partial \varphi}{\partial t}(t,x) - H^-\left(t,x,u,D\varphi,D^2\varphi\right), u(t,x) - h'(t,x)\right\}\right\} \geq 0;$$
3. *A function $u : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a viscosity solution of equation (11) if it is both a viscosity sub- and supersolution for this equation.*

Remark 8

(i) Of course, the definition of viscosity solutions for equation (12) is similar.

(ii) In the above definitions, one can take strict local maximum or minimum point instead of global maximum, respectively minimum point.

We recall (from [9], pp. 49) the definition of parabolic "superjet" and "subjet" of a function defined on a locally compact set, notions which will be needed during the proof of the uniqueness.

Definition 9 *For a function $u : [0,T) \times \mathbb{R}^n \longrightarrow \mathbb{R}$, the second-order parabolic superjet of u in $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$, denoted by $\mathcal{P}^{2,+}u(t_0,x_0)$, is the set of triplets $(p,q,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$ satisfying, as $(t,x) \to (t_0,x_0)$*
$$u(t,x) \leq u(t_0,x_0) + p(t-t_0) + \langle q, x-x_0\rangle + \frac{1}{2}\langle X(x-x_0), X(x-x_0)\rangle + o\left(|t-t_0| + |x-x_0|^2\right).$$
Switching the inequality sign in the above relation, we get the definition of the second-order parabolic subjet of u in (t_0,x_0), denoted by $\mathcal{P}^{2,-}u(t_0,x_0)$. It is clear that $\mathcal{P}^{2,-}u = -\mathcal{P}^{2,+}(-u)$.

One can give (see [9]) the definition of viscosity subsolution, resp. supersolution in terms of superjets, respectively subjets, as it follows:

Proposition 10 *An upper semicontinuous function $u : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying $u(T,\cdot) \leq g$, is a viscosity subsolution of equation (11) if and only if, for every $(t,x) \in [0,T) \times \mathbb{R}^n$ and every $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$, we have*
$$\min\left\{u(t,x) - h(t,x), \max\left\{-p - H^-(t,x,u(t,x),q,X), u(t,x) - h'(t,x)\right\}\right\} \leq 0.$$

A similar result holds for viscosity supersolutions.

Since H^- is continuous, one can replace the superjets and subjets with their closure, whose definition is given below:

Definition 11 *For $u : [0,T) \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$, we define $\bar{\mathcal{P}}^{2,+}u(t_0,x_0)$ as the set of triplets $(p,q,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$ for which there exists a sequence $(t_n,x_n,p_n,q_n,X_n) \in [0,T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$ such that $(p_n,q_n,X_n) \in \mathcal{P}^{2,+}u(t_n,x_n)$, for all $n \in \mathbb{N}$, and*
$$(t_n,x_n,u(t_n,x_n),p_n,q_n,X_n) \to (t_0,x_0,u(t_0,x_0),p,q,X).$$
Similarly, we define $\bar{\mathcal{P}}^{2,-}u(t_0,x_0)$.

In order to prove the uniqueness result, we will need additional properties imposed on the generator F. We suppose that F satisfies the following assumption: there exist a constant $C > 0$ and, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$, such that for all $(t,x,y,z,\alpha,\beta) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times A \times B$ we have

(H3)
$$\left|\left(F + \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\right)(t,x,y,z,\alpha,\beta)\right| \leq C(1+|z|^2),$$
$$\frac{\partial F}{\partial y}(t,x,y,z,\alpha,\beta) \leq C_\varepsilon + \varepsilon|z|^2.$$

3.2 Results

The framework is now set for the main part of this paper.

Theorem 12 (Existence) *Under the assumptions (H1), (H2), and (H3), the lower value function $\mathcal{W}$ defined by (9) is a viscosity solution of the Isaacs equation with two barriers (11), while the upper value function $\mathcal{V}$ defined by (10) is a viscosity solution of the Isaacs equation (12).*

Theorem 13 (Uniqueness) *Under the assumptions (H1), (H2) and (H3), if u is a bounded viscosity subsolution and v is a bounded viscosity supersolution of equation (11), then*

$$u(t,x) \leq v(t,x), \ \forall (t,x) \in [0,T] \times \mathbb{R}^n.$$

The same comparison principle holds for the Isaacs equation (12).

Remark 14 *If, in addition, the Isaacs' condition holds,* i.e.

$$H^-(t,x,u,q,X) = H^+(t,x,u,q,X),$$

for every $(t,x,u,q,X) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n$, then the two Isaacs equations coincide and it follows that the upper and the lower value functions are equal, which means that the corresponding reflected stochastic differential game has a value.

4 Proofs

We will focus our attention on the first Isaacs equation (11), the case of the equation (12) being treated in a similar manner.

4.1 Uniqueness

Let us consider a subsolution u, respectively a supersolution v, of equation (11).

First, we will make a change of variable, which preserves viscosity sub- and supersolutions and which transforms the equation into a Isaacs equation whose Hamiltonian will satisfy some kind of monotonicity.

We consider $\tilde{C} := \max(\|u\|_\infty, \|v\|_\infty) + 1$ and introduce the positive, increasing function ρ used in [25], pp. 583:

$$\rho : \mathbb{R} \longrightarrow (-(\ln\gamma)/\lambda, +\infty), \quad \rho(x) := \frac{1}{\lambda}\ln\left(\frac{e^{\lambda\gamma x}+1}{\gamma}\right),$$

for positive γ and λ satisfying $-(\ln\gamma)/\lambda \leq \tilde{C}$. We make the change of variable $\bar{u} := \rho^{-1}(e^{Kt}(u-\tilde{C}))$, with $K > 0$. Equation (11) becomes

$$(13) \quad \begin{cases} \min\left\{\rho(\bar{u}) - \rho(\bar{h}), \max\left\{\rho'(\bar{u})\left[-\frac{\partial\bar{u}}{\partial t} - \bar{H}^-(t,x,\bar{u},D\bar{u},D^2\bar{u})\right], \rho(\bar{u}) - \rho(\bar{h}')\right\}\right\} = 0; \\ \bar{u}(T,x) = \bar{g}(x), \end{cases}$$

where

$$\bar{H}^-(t,x,\bar{u},\bar{q},\bar{X}) = \sup_{\alpha\in A}\inf_{\beta\in B}\left[\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(t,x,\alpha,\beta)\bar{X}\right) + \langle b(t,x,\alpha,\beta),\bar{q}\rangle + \bar{F}(t,x,\bar{u},\bar{q}\sigma(t,x,\alpha,\beta),\alpha,\beta)\right],$$

with $\bar{F}$ defined by

$$\bar{F}(t,x,\bar{u},\bar{z},\alpha,\beta) = \frac{\rho''(\bar{u})}{\rho'(\bar{u})}|\bar{z}|^2 - K\frac{\rho(\bar{u})}{\rho'(\bar{u})} + \frac{e^{Kt}}{\rho'(\bar{u})}F(t,x,e^{-Kt}\rho(\bar{u})+\tilde{C}, e^{-Kt}\rho'(\bar{u})\bar{z},\alpha,\beta)$$

and

$$\bar{h}(t,x) := \rho^{-1}(e^{Kt}(h(t,x) - \tilde{C})),$$
$$\bar{h}'(t,x) := \rho^{-1}(e^{Kt}(h'(t,x) - \tilde{C})),$$
$$\bar{g}(x) := \rho^{-1}(e^{Kt}(g(x) - \tilde{C})).$$

The function $\bar{F}$ verifies, for γ big enough,

Lemma 15 *There exist some positive constants $\tilde{K}$ and $\bar{C}$ such that for all $t \in (0,T)$, $(\alpha,\beta) \in A \times B$, $x, y \in \mathbb{R}^n$, $z, z' \in \mathbb{R}^d$, and $u, v \in \mathbb{R}$ such that $u < v$,*

$$\bar{F}(t,x,u,z,\alpha,\beta) - \bar{F}(t,y,v,z',\alpha,\beta) \le \mathcal{K}(z,z')\left(-\tilde{K}(u-v) + \bar{C}|x-y| + \bar{C}|z-z'|\right),$$

where $\mathcal{K}(z,z') := \left(1 + \frac{|z|^2}{2} + \frac{|z'|^2}{2}\right)$.

As we said before, by this transformation, if u (resp., v) is a subsolution (resp., a supersolution) of equation (11), then $\bar{u}$ (resp., $\bar{v}$) is one for equation (13). Therefore, we want to prove that

$$M := \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} (\bar{u}(t,x) - \bar{v}(t,x))$$

is negative. Define also

$$M(h) := \sup_{|x-y|\le h} |\bar{u}(t,x) - \bar{v}(t,y)| \quad \text{and} \quad M' := \lim_{h\to 0} M(h).$$

It is clear that $M \le M'$.

We assume to the contrary $M > 0$ and define for every ε, $\eta > 0$

$$\Psi_{\varepsilon,\eta}(t,x,y) := \bar{u}(t,x) - \bar{v}(t,y) - \frac{|x-y|^2}{\varepsilon^2} - \eta(|x|^2 + |y|^2).$$

Let us consider

$$M_{\varepsilon,\eta} := \sup_{(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n} \Psi_{\varepsilon,\eta}(t,x,y) = \max_{(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n} \Psi_{\varepsilon,\eta}(t,x,y)$$

Since the functions $\bar{u}$ and $\bar{v}$ are bounded, the supremum of $\Psi_{\varepsilon,\eta}$ is reached at some point $(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, y^{\varepsilon,\eta})$, which will be denoted for simplicity $\left(\hat{t}, \hat{x}, \hat{y}\right)$. We will use this notations each time we do not want to make explicit the dependence on ε and η.

Let recall some notations and results from [25], pp. 586-587: for the sequence $(a_{\varepsilon,\eta})_{\varepsilon,\eta}$, we write $a = \lim_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta}$ if

$$\liminf\nolimits_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta} = \limsup\nolimits_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta} = a,$$

where

$$\liminf\nolimits_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta} = \liminf\nolimits_{\eta\to 0}\left(\liminf\nolimits_{\varepsilon\to 0} a_{\varepsilon,\eta}\right),$$
$$\limsup\nolimits_{\varepsilon\ll\eta\to 0} a_{\varepsilon,\eta} = \limsup\nolimits_{\eta\to 0}\left(\limsup\nolimits_{\varepsilon\to 0} a_{\varepsilon,\eta}\right).$$

The following result is the equivalent of Lemma 3.1 of [9].

Lemma 16 *Considering the above notations, we have*

$$(i) \qquad \lim_{\varepsilon\ll\eta\to 0} M_{\varepsilon,\eta} = M, \quad \lim_{\varepsilon\ll\eta\to 0} \bar{u}\left(\hat{t},\hat{x}\right) - \bar{v}\left(\hat{t},\hat{y}\right) = M;$$
$$(ii) \qquad \lim_{\eta\ll\varepsilon\to 0} M_{\varepsilon,\eta} = M', \quad \lim_{\eta\ll\varepsilon\to 0} \bar{u}\left(\hat{t},\hat{x}\right) - \bar{v}\left(\hat{t},\hat{y}\right) = M';$$
$$(iii) \qquad \lim_{\eta\ll\varepsilon\to 0} \frac{|\hat{x}-\hat{y}|}{\varepsilon} = 0, \quad \lim_{\eta\ll\varepsilon\to 0} \eta(|\hat{x}|^2 + |\hat{y}|^2) = 0.$$

So, by extracting a subsequence, we suppose that for every η, the sequence $(t^{\varepsilon,\eta})_\varepsilon$ converges to a limit t^η as $\varepsilon \to 0$ and, extracting again a subsequence, the sequences $(x^{\varepsilon,\eta})_\varepsilon$ and $(y^{\varepsilon,\eta})_\varepsilon$ converge to a common limit x^η.

Now define the functions $\phi_1,\ \phi_2 : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\phi_1(t,x) = \bar{v}(t,y^{\varepsilon,\eta}) + \frac{|x-y^{\varepsilon,\eta}|^2}{\varepsilon^2} + \eta(|x|^2 + |y^{\varepsilon,\eta}|^2)$$

$$\phi_2(t,y) = \bar{u}(t,x^{\varepsilon,\eta}) - \frac{|x^{\varepsilon,\eta}-y|^2}{\varepsilon^2} - \eta(|x^{\varepsilon,\eta}|^2 + |y|^2).$$

It is obvious that $(t^{\varepsilon,\eta}, x^{\varepsilon,\eta})$ is a maximum point for the function $(t,x) \longmapsto \Psi_{\varepsilon,\eta}(t,x,y^{\varepsilon,\eta}) = (\bar{u}-\phi_1)(t,x)$, while $(t^{\varepsilon,\eta}, y^{\varepsilon,\eta})$ is a minimum point for $(t,y) \longmapsto -\Psi_{\varepsilon,\eta}(t,x^{\varepsilon,\eta},y) = (\phi_2 - \bar{v})(t,y)$. Since $\bar{u}$ and $\bar{v}$ are viscosity subsolution, respectively supersolution, we obtain

- either $t^{\varepsilon,\eta} = T$ and then $\bar{u}(T,x^{\varepsilon,\eta}) \le \bar{g}(x^{\varepsilon,\eta})$ and $\bar{g}(y^{\varepsilon,\eta}) \le \bar{v}(T,y^{\varepsilon,\eta})$,
- or $t^{\varepsilon,\eta} \neq T$ and we have, in $(\hat{t},\hat{x})$,

$$\min\left\{\rho(\bar{u}) - \rho(\bar{h}), \max\left\{\rho'(\bar{u})\left[-\frac{\partial\phi_1}{\partial t} - \bar{H}^-\left(\hat{t},\hat{x},\bar{u},D\phi_1,D^2\phi_1\right)\right], \rho(\bar{u}) - \rho(\bar{h}')\right\}\right\} \le 0$$

  and, respectively, in $(\hat{t},\hat{y})$,

$$\min\left\{\rho(\bar{v}) - \rho(\bar{h}), \max\left\{\rho'(\bar{v})\left[-\frac{\partial\phi_2}{\partial t} - \bar{H}^-\left(\hat{t},\hat{y},\bar{v},D\phi_2,D^2\phi_2\right)\right], \rho(\bar{v}) - \rho(\bar{h}')\right\}\right\} \ge 0.$$

In the first situation, there exists a subsequence of $(t^\eta)_\eta$, supposed, without restricting the generality, to be the same, such that $t^\eta = T$ for every η. The semicontinuity of the functions $\bar{u}$ and $\bar{v}$ and the continuity of $\bar{g}$ give us that, for all η and ε sufficiently small,

$$\bar{u}(\hat{t},\hat{x}) \le \bar{u}(T,x^\eta) + \eta \le \bar{g}(x^\eta) + \eta \quad \text{and} \quad \bar{g}(x^\eta) - \eta \le \bar{v}(T,x^\eta) - \eta \le \bar{v}(\hat{t},\hat{x}).$$

So, $\bar{u}\left(\hat{t},\hat{x}\right) \le \bar{v}\left(\hat{t},\hat{x}\right) + 2\eta$ and, from here, we find that $M_{\varepsilon,\eta} \le 2\eta$. Taking the limit of ε, and after this η, to zero, it follows that $M \le 0$, which is a contradiction. Therefore $t^{\varepsilon,\eta} \neq T$.

Now, since ρ is an increasing function, if there exists a subsequence $t^\eta \neq T$, and then, for every η a subsequence of $(x^{\varepsilon,\eta})_\varepsilon$ and one of $(y^{\varepsilon,\eta})_\varepsilon$ such that the following inequalities hold

$$\bar{u}(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - \bar{h}(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) \le 0 \quad \text{and} \quad \bar{v}(t^{\varepsilon,\eta},y^{\varepsilon,\eta}) \ge \bar{h}(t^{\varepsilon,\eta},y^{\varepsilon,\eta}),$$

we obtain

$$M_{\varepsilon,\eta} \le \bar{u}(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - \bar{v}(t^{\varepsilon,\eta},y^{\varepsilon,\eta}) \le \bar{h}(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - \bar{h}(t^{\varepsilon,\eta},y^{\varepsilon,\eta}).$$

The continuity of $\bar{h}$ and the above Lemma leads again to a contradiction. When, also on a subsequence like in the previous case, $\bar{u}(\hat{t},\hat{x}) \le \bar{h}'(\hat{t},\hat{x})$,

$$-\frac{\partial\phi_1}{\partial t}\left(\hat{t},\hat{x}\right) - \bar{H}^-\left(\hat{t},\hat{x},\bar{u}\left(\hat{t},\hat{x}\right),D\phi_1\left(\hat{t},\hat{x}\right),D^2\phi_1\left(\hat{t},\hat{x}\right)\right) \le 0$$

and $\bar{v}(\hat{t},\hat{y}) \ge \bar{h}'(\hat{t},\hat{y})$, we obtain also that $M \le 0$.

For the remaining situation, we first use Theorem 8.3 from [9].

Setting $\varphi(x,y) := \dfrac{|x-y|^2}{\varepsilon^2} + \eta(|x|^2 + |y|^2)$, from the mentioned result we have that there exist the matrices $\bar{X}, \bar{Y} \in \mathbb{S}_n$ such that

$$\begin{cases} (0, D_x\varphi(\hat{x},\hat{y}), \bar{X}) \in \bar{\mathcal{P}}^{2,+}\bar{u}(\hat{t},\hat{x}), \\ (0, D_y\varphi(\hat{x},\hat{y}), -\bar{Y}) \in \bar{\mathcal{P}}^{2,+}(-\bar{v})(\hat{t},\hat{y}) = -\bar{\mathcal{P}}^{2,-}\bar{v}(\hat{t},\hat{y}). \end{cases}$$

So $\left(0, -D_y\varphi\left(\hat{t}, \hat{x}, \hat{y}\right), \bar{Y}\right) \in \bar{\mathcal{P}}^{2,-}\bar{v}(\hat{t}, \hat{y})$.

Moreover,

$$\begin{pmatrix} \bar{X} & 0 \\ 0 & -\bar{Y} \end{pmatrix} \leq \frac{2}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\eta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{14}$$

We have then from the definition of the sub- and superjet that

$$\begin{cases} \bar{H}^-\left(\hat{t}, \hat{x}, \bar{u}\left(\hat{t}, \hat{x}\right), D_x\varphi\left(\hat{x}, \hat{y}\right), \bar{X}\right) \geq 0 \text{ and} \\ \bar{H}^-\left(\hat{t}, \hat{y}, \bar{v}\left(\hat{t}, \hat{y}\right), -D_y\varphi\left(\hat{x}, \hat{y}\right), \bar{Y}\right) \leq 0, \end{cases}$$

which implies

$$\bar{H}^-\left(\hat{t}, \hat{x}, \bar{u}(\hat{t}, \hat{x}), \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} + 2\eta\hat{x}, \bar{X}\right) \geq \bar{H}^-\left(\hat{t}, \hat{y}, \bar{v}(\hat{t}, \hat{y}), \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} - 2\eta\hat{y}, \bar{Y}\right).$$

By denoting

$$\begin{aligned} \hat{\mathcal{K}}_x^{\alpha,\beta} &:= \frac{1}{2}Tr(\sigma\sigma^T(\hat{t}, \hat{x}, \alpha, \beta)\bar{X}) \\ &+ \left\langle b\left(\hat{t}, \hat{x}, \alpha, \beta\right), \frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2} + 2\eta\hat{x}\right\rangle + \bar{F}\left(\hat{t}, \hat{x}, \bar{u}\left(\hat{t}, \hat{x}\right), \left(\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2} + 2\eta\hat{x}\right)\sigma\left(\hat{t}, \hat{x}, \alpha, \beta\right), \alpha, \beta\right) \end{aligned}$$

and

$$\begin{aligned} \hat{\mathcal{K}}_y^{\alpha,\beta} &:= \frac{1}{2}Tr(\sigma\sigma^T(\hat{t}, \hat{y}, \alpha, \beta)\bar{Y}) \\ &+ \left\langle b\left(\hat{t}, \hat{y}, \alpha, \beta\right), \frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2} - 2\eta\hat{y}\right\rangle + \bar{F}\left(\hat{t}, \hat{y}, \bar{v}\left(\hat{t}, \hat{y}\right), \left(\frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2} - 2\eta\hat{y}\right)\sigma\left(\hat{t}, \hat{y}, \alpha, \beta\right), \alpha, \beta\right) \end{aligned}$$

this can be written as

$$\sup_{\alpha\in A}\inf_{\beta\in B}\hat{\mathcal{K}}_x^{\alpha,\beta} \geq \sup_{\alpha\in A}\inf_{\beta\in B}\hat{\mathcal{K}}_y^{\alpha,\beta}. \tag{15}$$

Also, by denoting

$$\begin{cases} \hat{b}_x := b(\hat{t}, \hat{x}, \alpha, \beta), & \hat{b}_y := b(\hat{t}, \hat{y}, \alpha, \beta), \\ \hat{\sigma}_x := \sigma(\hat{t}, \hat{x}, \alpha, \beta), & \hat{\sigma}_y := \sigma(\hat{t}, \hat{y}, \alpha, \beta), \\ \bar{q}_x := \dfrac{2(\hat{x}-\hat{y})}{\varepsilon^2} + 2\eta\hat{x}, & \bar{q}_y := \dfrac{2(\hat{x}-\hat{y})}{\varepsilon^2} - 2\eta\hat{y}, \end{cases}$$

the inequality (14) infers that

$$\frac{1}{2}Tr(\hat{\sigma}_x\hat{\sigma}_x^T\bar{X}) \leq \frac{1}{2}Tr(\hat{\sigma}_y\hat{\sigma}_y^T\bar{Y}) + \frac{1}{\varepsilon^2}|\hat{\sigma}_x - \hat{\sigma}_y|^2 + \eta(|\hat{\sigma}_x|^2 + |\hat{\sigma}_y|^2)$$

On the other hand

$$\left\langle\hat{b}_x, \bar{q}_x\right\rangle = \left\langle\hat{b}_y, \bar{q}_y\right\rangle + \left\langle\hat{b}_x - \hat{b}_y, \frac{2\left(\hat{x}-\hat{y}\right)}{\varepsilon^2}\right\rangle + \left\langle\hat{b}_y, 2\eta\hat{y}\right\rangle + \left\langle\hat{b}_x, 2\eta\hat{x}\right\rangle$$

and

$$\begin{aligned} \bar{F}\left(\hat{t}, \hat{x}, \bar{u}\left(\hat{t}, \hat{x}\right), \bar{q}_x\hat{\sigma}_x, \alpha, \beta\right) &\leq \bar{F}\left(\hat{t}, \hat{y}, \bar{v}\left(\hat{t}, \hat{y}\right), \bar{q}_y\hat{\sigma}_y, \alpha, \beta\right) \\ &+ \mathcal{K}\left(\bar{q}_x\hat{\sigma}_x, \bar{q}_y\hat{\sigma}_y\right)\left(-\tilde{K}\left(\bar{u}\left(\hat{t}, \hat{x}\right) - \bar{v}\left(\hat{t}, \hat{y}\right)\right) + \bar{C}\left|\hat{x}-\hat{y}\right| + \bar{C}\left|\bar{q}_x\hat{\sigma}_x - \bar{q}_y\hat{\sigma}_y\right|\right). \end{aligned}$$

Adding the last three relations, we obtain, for all $(\alpha,\beta)\in A\times B$

$$\begin{aligned}\mathcal{K}_x^{\alpha,\beta} &\le \mathcal{K}_y^{\alpha,\beta}+\frac{C_L^2}{\varepsilon^2}|\hat{x}-\hat{y}|^2+C\eta(1+|\hat{x}|^2+|\hat{y}|^2)+\frac{2C_L}{\varepsilon^2}|\hat{x}-\hat{y}|^2\\ &+\mathcal{K}\left(\bar{q}_x\hat{\sigma}_x,\bar{q}_y\hat{\sigma}_y\right)\left(-\tilde{K}M_{\varepsilon,\eta}+\bar{C}|\hat{x}-\hat{y}|+2\bar{C}\left(C_L\frac{|\hat{x}-\hat{y}|^2}{\varepsilon^2}+\eta(|\hat{\sigma}_x\hat{x}|+|\hat{\sigma}_y\hat{y}|)\right)\right).\end{aligned}$$

Taking the $\sup_{\alpha\in A}\inf_{\beta\in B}$, and then passing to the limit as $\varepsilon\to 0$ and $\eta\to 0$ (in this order), we obtain

$$\sup_{\alpha\in A}\inf_{\beta\in B}\hat{\mathcal{K}}_x^{\alpha,\beta}\le\sup_{\alpha\in A}\inf_{\beta\in B}\hat{\mathcal{K}}_y^{\alpha,\beta}-\tilde{K}M,$$

which contradicts (15) if M is strictly positive as we supposed. So M must be less or equal to zero and we obtain the desired comparison result.

4.2 Existence

As indicated in preliminaries, we make the following transform:

$$f(t,x,y,z,\alpha,\beta):=\begin{cases}2Cy\left[F\left(t,x,\frac{\ln y}{2C},\frac{z}{2Cy},\alpha,\beta\right)-\frac{|z|^2}{4Cy^2}\right], & y>0\\ 0, & y\le 0\end{cases}$$

for $(\alpha,\beta)\in A\times B$ and $(t,\omega,y,z)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$, where C is the constant from (H3). We also set, for the simplicity of notations,

$$\bar{h}:=\exp(2Ch),\ \bar{h}':=\exp(2Ch'),\ \bar{g}:=\exp(2Cg)\ \text{ and } M:=\max\left\{\inf\left(1/\bar{h}\right),\sup\bar{h}'\right\}.$$

As showed in [17], under conditions (H1), (H2) and (H3), the RBSDE associated with

$$(f,\bar{g}(X_T^{t,x;\alpha,\beta}),\bar{h}(\cdot,X^{t,x;\alpha,\beta}),\bar{h}'(\cdot,X^{t,x;\alpha,\beta}))$$

has a maximal solution, denoted $\left(y^{t,x;\alpha,\beta},z^{t,x;\alpha,\beta},k^{+,t,x;\alpha,\beta},k^{-,t,x;\alpha,\beta}\right)$. In order to use already known results on the lower value function associated with a RBSDE with double barrier with a Lipschitz generator (see [8]), we approximate, in a monotone manner, the function f with Lipschitz functions. For that, we use the method considered in [25], which we describe in the sequel.

We will make use, for $p\ge 0$, of functions $\rho_p\in C^\infty(\mathbb{R})$ satisfying $\rho_p=1$ on $[-p,p]$ and $\rho_p=0$ on $[-(p+1),p+1]^{\mathrm{c}}$. Let us consider the function

$$\tilde{f}(t,x,y,z,\alpha,\beta):=f(t,x,y,z,\alpha,\beta)\,\rho_M\left(\frac{\ln y}{2C}\right),$$

which is bounded by a constant $C'>0$. Let, for $p\in\mathbb{N}$,

$$\tilde{f}^p(t,x,y,z,\alpha,\beta):=\tilde{f}(t,x,y,z,\alpha,\beta)\,\rho_p(|x|+|z|)+\frac{3}{2^{p+2}};$$

it is clear that the function $\tilde{f}^p$ is bounded and uniformly continuous.

Now, by a standard *mollification* procedure, we approximate $\tilde{f}^p$ by Lipschitz functions. Let $\theta\in C^\infty\left(\mathbb{R}^{n+1+d}\right)$ satisfying $\theta\ge 0$, $\operatorname{supp}\theta\subseteq B(0;1)$ and $\int_{\mathbb{R}^{n+1+d}}\theta(a)\,da=1$; we set

$$\tilde{f}_\varepsilon^p(t,x,y,z,\alpha,\beta):=\frac{1}{\varepsilon^{n+1+d}}\int_{\mathbb{R}^{n+1+d}}\theta\left(\frac{x-x'}{\varepsilon},\frac{y-y'}{\varepsilon},\frac{z-z'}{\varepsilon}\right)\tilde{f}^p(t,x,y,z,\alpha,\beta)\,dx'dy'dz'.$$

Then $\tilde{f}^p_\varepsilon$ is Lipschitz in $(x,y,z)\in\mathbb{R}^{n+1+d}$ and

$$|\tilde{f}^p_\varepsilon-\tilde{f}_p|\le\eta_{\tilde{f}_p}(\varepsilon),$$

where $\eta_{\tilde{f}^p}$ is the modulus of uniform continuity of $\tilde{f}^p$. Hence, one can extract a sequence $\varepsilon_p\searrow 0$ such that

$$|\tilde{f}^p_{\varepsilon_p}-\tilde{f}_p|\le 2^{-(p+2)},\ \forall p\in\mathbb{N}^*.$$

An easy calculus shows us that $\tilde{f}^p_{\varepsilon_p}$ is upper bounded by $C'+2^{-p}$ and

$$(\tilde{f}^{p+1}_{\varepsilon_{p+1}}-\tilde{f}^p_{\varepsilon_p})(t,x,y,z,\alpha,\beta)\le\tilde{f}(t,x,y,z,\alpha,\beta)(\rho_{p+1}-\rho_p)(|x|+|z|).$$

If we set, for $p\in\mathbb{N}^*$,

$$f^p(t,x,y,z,\alpha,\beta):=\rho_{p-1}(|x|+|z|)\,\tilde{f}^p_{\varepsilon_p}(t,x,y,z,\alpha,\beta)+(1-\rho_{p-1}(|x|+|z|))\left(C'+2^{-p}\right),$$

then the functions f^p are still Lipschitz in $(x,y,z)\in\mathbb{R}^{n+1+d}$ and $f^p\searrow\tilde{f}$, the convergence being uniform.

Consider now, for $p\in\mathbb{N}^*$, the RBSDE associated with $(f^p,\bar{g}(X^{t,x;\alpha,\beta}_T),\bar{h}(\cdot,X^{t,x;\alpha,\beta}),\bar{h}'(\cdot,X^{t,x;\alpha,\beta}))$. According to [16], Theorem 3.7, it has a unique solution in $\mathcal{S}^2\times\mathcal{L}^{2,d}\times\mathcal{M}\times\mathcal{M}$, denoted

$$\left(y^{p;t,x;\alpha,\beta},z^{p;t,x;\alpha,\beta},k^{+,p;t,x;\alpha,\beta},k^{-,p;t,x;\alpha,\beta}\right).$$

By the comparison result, it is obvious that the sequence $\left(y^{p;t,x;\alpha,\beta}\right)_{p\in\mathbb{N}^*}$ is non-increasing.

Proposition 17 *For every $(t,x)\in[0,T]$, $(\alpha,\beta)\in\mathcal{A}_t\times\mathcal{B}_t$, $\lim_{p\to\infty}y^{p;t,x;\alpha,\beta}_s=y^{t,x;\alpha,\beta}_s$, for all $s\in[t,T]$, $\mathbb{P}$-a.s.*

The proof of this result follows the same five steps of the proof of the Theorem 3.1 in [17] since the different approximation sequence is constituted also by generators which are bounded and continuous. So we skip the proof.

Let us consider, for any $p\in\mathbb{N}^*$, and any given control processes $\alpha(\cdot)\in\mathcal{A}_t$, $\beta(\cdot)\in\mathcal{B}_t$, the associated cost functional

$$j^p(t,x;\alpha,\beta):=y^{p;t,x;\alpha,\beta}_t,\quad (t,x)\in[0,T]\times\mathbb{R}^n$$

and define the lower value function of the approximative stochastic differential game

$$w^p(t,x):=\operatorname*{essinf}_{S_2\in\mathbb{B}_t}\operatorname*{esssup}_{\alpha\in\mathcal{A}_t}j^p(t,x;\alpha,S_2(\alpha)).$$

It is known, from [8], Proposition 3.1, that w^p is deterministic and is the unique viscosity solution of the equation

(16)
$$\begin{cases}\min\left\{u(t,x)-\bar{h}(t,x),\max\left\{-\dfrac{\partial u}{\partial t}(t,x)-H^p\left(t,x,u,Du,D^2u\right),u(t,x)-\bar{h}'(t,x)\right\}\right\}=0;\\ u(T,x)=\bar{g}(x),\end{cases}$$

where

$$H^p(t,x,u,q,X)$$
$$:=\sup_{\alpha\in A}\inf_{\beta\in B}\left\{\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(t,x,\alpha,\beta)X\right)+\langle b(t,x,\alpha,\beta),q\rangle+f^p(t,x,u,q\sigma(t,x,\alpha,\beta),\alpha,\beta)\right\}.$$

On the other hand, since the processes $\left(y^{p;t,x;\alpha,\beta}\right)_{p\in\mathbb{N}^*}$ form a non-increasing sequence, the sequence $(w^p)_{p\in\mathbb{N}^*}$ is also non-increasing. By boundedness, it has a limit, $w^0:[0,T]\times\mathbb{R}^n\longrightarrow\mathbb{R}$.

Proposition 18 *The function w^0 is a viscosity subsolution of the equation*

$$(17)\quad \begin{cases} \min\left\{u(t,x)-\bar{h}(t,x), \max\left\{-\dfrac{\partial u}{\partial t}(t,x)-\bar{H}\left(t,x,u,Du,D^2u\right), u(t,x)-\bar{h}'(t,x)\right\}\right\}=0; \\ u(T,x)=\bar{g}(x), \end{cases}$$

where

$$\bar{H}(t,x,u,q,X)$$
$$:= \sup_{\alpha\in A}\inf_{\beta\in B}\left\{\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(t,x,\alpha,\beta)X\right)+\langle b(t,x,\alpha,\beta),q\rangle+f(t,x,u,q\sigma(t,x,\alpha,\beta),\alpha,\beta)\right\}.$$

Proof. It is clear that w^0 is an upper semicontinuous function satisfying $w^0(T,\cdot)\le g(\cdot)$. Let now suppose that $\varphi\in C^{1,2}([0,T)\times\mathbb{R}^n)$ and that $(t,x)\in[0,T)\times\mathbb{R}^n$ is a strict local maximum point for $w^0-\varphi$. Then there exists a sequence (t_p,x_p) in $[0,T)\times\mathbb{R}^n$, converging to (t,x), such that $w^p-\varphi$ has a local maximum point in (t_p,x_p) for all $p\in\mathbb{N}^*$ and $\lim_{p\to\infty}w^p(t_p,x_p)=w^0(t,x)$.

Since w^p is a viscosity solution for equation (16), it follows that, for all $p\in\mathbb{N}^*$, in (t_p,x_p),

$$\min\left\{w^p-\bar{h}, \max\left\{-\frac{\partial\varphi}{\partial t}-H^p\left(t_p,x_p,w^p,D\varphi,D^2\varphi\right), w^p-\bar{h}'\right\}\right\}=0.$$

Because $\bar{h}(t,x)\le w^0(t,x)\le\bar{h}'(t,x)$, we analyse just two cases:

(i) $w^0(t,x)=\bar{h}(t,x)$, hence equation (17) is trivially satisfied;

(ii) $w^0(t,x)>\bar{h}(t,x)$, which implies $w^p(t_p,x_p)>\bar{h}(t_p,x_p)$ for sufficiently large p, and so

$$-\frac{\partial\varphi}{\partial t}(t_p,x_p)-H^p\left(t_p,x_p,w^p(t_p,x_p),D\varphi(t_p,x_p),D^2\varphi(t_p,x_p)\right)\le 0.$$

Since

$$\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(t_p,x_p,\alpha,\beta)D^2\varphi(t_p,x_p)\right)$$
$$+\langle b(t_p,x_p,\alpha,\beta),D\varphi(t_p,x_p)\rangle+f^p(t_p,x_p,w^p(t_p,x_p),D\varphi(t_p,x_p)\sigma(t_p,x_p,\alpha,\beta),\alpha,\beta)$$

converges uniformly (with respect to α and β) to

$$\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^T(t,x,\alpha,\beta)D^2\varphi(t,x)\right)$$
$$+\langle b(t,x,\alpha,\beta),D\varphi(t,x)\rangle+\tilde{f}\left(t,x,w^0(t,x),D\varphi(t,x)\sigma(t,x,\alpha,\beta),\alpha,\beta\right),$$

and $w^0(t,x)\le M$, it follows that

$$-\frac{\partial\varphi}{\partial t}(t,x)-\bar{H}\left(t,x,w^0(t,x),D\varphi(t,x),D^2\varphi(t,x)\right)\le 0.$$

This finishes our proof. ■

One can repeat the above schema, but with lower approximation, *i.e.* we can construct (in the same manner), an increasing sequence of Lipschitz, bounded functions (f_p) converging to $\tilde{f}$. By denoting

$$(y_p^{t,x;\alpha,\beta}, z_p^{t,x;\alpha,\beta}, k_p^{+,t,x;\alpha,\beta}, k_p^{-,t,x;\alpha,\beta})$$

the solution of the RBSDE associated with $(f_p, \bar{g}(X_T^{t,x;\alpha,\beta}), \bar{h}(\cdot, X^{t,x;\alpha,\beta}), \bar{h}'(\cdot, X^{t,x;\alpha,\beta}))$, one can show that $\left(y_p^{t,x;\alpha,\beta}\right)$ is an increasing sequence of processes, converging to a minimal solution of the RBSDE associated with $(f, \bar{g}(X_T^{t,x;\alpha,\beta}), \bar{h}(\cdot, X^{t,x;\alpha,\beta}), \bar{h}'(\cdot, X^{t,x;\alpha,\beta}))$. Let us denote

$$j_p\left(t,x;\alpha,\beta\right) := y_{p,t}^{t,x;\alpha,\beta}, \quad (t,x) \in [0,T] \times \mathbb{R}^n$$

and

$$w_p\left(t,x\right) := \operatorname*{essinf}_{S_2 \in \mathbb{B}_t} \operatorname*{esssup}_{\alpha \in \mathcal{A}_t} j_p\left(t,x;\alpha, S_2\left(\alpha\right)\right).$$

Then, setting $w_0 := \lim w_p$, we have the analogous of Proposition 18, whose proof is essentially the same.

Proposition 19 *The function w_0 is a viscosity supersolution of the equation (17).*

Let us now define $\mathcal{W}^0 := \ln \dfrac{w^0}{2C}$ and $\mathcal{W}_0 := \ln \dfrac{w_0}{2C}$. It is straightforward to show that $\mathcal{W}^0$ and $\mathcal{W}_0$ are viscosity subsolution, respectively supersolution, of equation (11). By the comparison result (see [17], Remark 3.3), $\mathbb{P}$-a.s.,

$$\mathcal{W}^0\left(t,x\right) \geq \mathcal{W}\left(t,x\right) \geq \mathcal{W}_0\left(t,x\right), \ \forall \left(t,x\right) \in [0,T] \times \mathbb{R}^n,$$

since, for all $p \in \mathbb{N}^*$, $(t,x) \in [0,T] \times \mathbb{R}^n$ and $(\alpha\left(\cdot\right), \beta\left(\cdot\right)) \in \mathcal{A}_t \times \mathcal{B}_t$, $\mathbb{P}$-a.s.,

$$y_s^{p;t,x;\alpha,\beta} \geq y_s^{t,x;\alpha,\beta} \geq y_{p,s}^{t,x;\alpha,\beta}, \ \forall s \in [t,T].$$

On the other hand, by Theorem 13, we have that $\mathcal{W}^0 \leq \mathcal{W}_0$. These inequalities imply that $\mathcal{W}$ is deterministic and is a viscosity solution of equation (11). The existence result is thus proved.

Remark 20 *As one can see from the proof of the existence result, that $\mathcal{W}$ can be defined via any solution of the BSDE associated with (F, g, h, h'), not necessarily the maximal one. This could be used to deduce the uniqueness for this equation.*

5 Applications

As we said in the introduction, one application comes from financial markets. Indeed, it is possible to use our framework for the study of American game options and of the Ramsey's model. We now focus on the link between the payoffs Γ defined in (2) and the solutions of the reflected BSDEs associated with (7).

Proposition 21 *Assume that $F(t,x,y,z,\alpha,\beta) = \varphi(t,x,\alpha,\beta) + \frac{1}{2}|z|^2$, where φ is a bounded measurable function. For any stopping times τ and σ let us consider the following standard BSDE:*

$$(18) \qquad \begin{cases} (Y^{(t,x);(\alpha,\sigma),(\beta,\tau)}, Z^{(t,x);(\alpha,\sigma),(\beta,\tau)}) \in \mathcal{S}^2 \times \mathcal{H}^{2,d} \\ Y_s^{(t,x);(\alpha,\sigma),(\beta,\tau)} = h(X_\sigma^{t,x;\alpha,\beta}) 1_{[\sigma \leq \tau < T]} + h'(X_\tau^{t,x;\alpha,\beta}) 1_{[\tau < \sigma]} + g(X_T^{t,x;\alpha,\beta}) 1_{[\sigma = \tau = T]} \\ \qquad + \displaystyle\int_{s\wedge\tau\wedge\sigma}^{T\wedge\tau\wedge\sigma} \left\{ \varphi(r, X_r^{t,x;\alpha,\beta}, \alpha_r, \beta_r) + \frac{1}{2} |Z_r^{(t,x);(\alpha,\sigma),(\beta,\tau)}|^2 \right\} dr \\ \qquad - \displaystyle\int_{s\wedge\tau\wedge\sigma}^{T\wedge\tau\wedge\sigma} Z_r^{(t,x);(\alpha,\sigma),(\beta,\tau)} dW_r, \ \forall s \leq T. \end{cases}$$

Then

$$\Gamma(\alpha,\sigma;\beta,\tau) = \exp\{Y_0^{(0,x);(\alpha,\sigma),(\beta,\tau)}\},$$

$$\operatorname*{essinf}_{S_2 \in \mathbb{B}_t} \operatorname*{esssup}_{\alpha \in \mathcal{A}_t} Y_t^{t,x;\alpha,S_2(\alpha)} = \operatorname*{essinf}_{S_2 \in \mathbb{B}_t} \operatorname*{esssup}_{\alpha \in \mathcal{A}_t} \operatorname*{essinf}_{\tau \in \mathcal{T}_t} \operatorname*{esssup}_{\sigma \in \mathcal{T}_t} Y_t^{(t,x);(\alpha,\sigma),(S_2(\alpha),\tau)}$$

and

$$\operatorname*{esssup}_{S_1 \in \mathbb{A}_t} \operatorname*{essinf}_{\beta \in \mathcal{B}_t} Y_t^{t,x;S_1(\beta),\beta} = \operatorname*{esssup}_{S_1 \in \mathbb{A}_t} \operatorname*{essinf}_{\beta \in \mathcal{B}_t} \operatorname*{esssup}_{\sigma \in \mathcal{T}_t} \operatorname*{essinf}_{\tau \in \mathcal{T}_t} Y_t^{(t,x);(S_1(\beta),\sigma),(\beta,\tau)},$$

where by $\mathcal{T}_t$ we denoted the set of stopping times τ such that $t \le \tau \le T$.

Proof. For the sake of simplicity we denote $Y^{(t,x);(\alpha,\sigma),(\beta,\tau)}$ by Y. First note that since the functions h, h', g and φ are bounded then through the result by Kobylanski [25], Theorem 2.3, the solution of (18) exists and is unique. Now for $s \le T$ let us set

$$\tilde{Y}_s = \exp\left\{Y_s + \int_0^{s\wedge\tau\wedge\sigma} \varphi(r, X_r^{t,x;\alpha,\beta}, \alpha_r, \beta_r)dr\right\}$$

Using now Itô's formula to obtain that:

$$\begin{cases} d\tilde{Y}_s = \tilde{Z}_s dW_s, \ s \le T \text{ and} \\ \tilde{Y}_T = \exp\Big\{h(X_\sigma^{t,x;\alpha,\beta})1_{[\sigma\le\tau<T]} + h'(\tau, X_\tau^{t,x;\alpha,\beta})1_{[\tau<\sigma]} + g(X_T^{t,x;\alpha,\beta})1_{[\sigma=\tau=T]} \\ \qquad\qquad\qquad + \displaystyle\int_0^{T\wedge\tau\wedge\sigma} \varphi(r, X_r^{t,x;\alpha,\beta}, \alpha_r, \beta_r)dr\Big\}. \end{cases}$$

It implies that $\mathbb{E}[\tilde{Y}_0] = \mathbb{E}[\tilde{Y}_T]$. As $\tilde{Y}_0$ is deterministic since it is $\mathcal{F}_0$-measurable then $\tilde{Y}_0 = \mathbb{E}[\tilde{Y}_0] = \mathbb{E}[\tilde{Y}_T]$, i.e. $\exp\{Y_0\} = \Gamma(\alpha, \sigma; \beta, \tau)$.

Let us now deal with the second relations. First note (see for example [13], Proposition 2.3) that the characterization of a solution for a BSDE with two reflecting barriers implies that for any α and β,

$$Y_t^{t,x;\alpha,\beta} = \operatorname*{essinf}_{\tau\ge t} \operatorname*{esssup}_{\sigma\ge t} Y_t^{(t,x);(\alpha,\sigma),(\beta,\tau)} = \operatorname*{esssup}_{\sigma\ge t} \operatorname*{essinf}_{\tau\ge t} Y_t^{(t,x);(\alpha,\sigma),(\beta,\tau)} \tag{19}$$

since

$$\exp(Y_t^{t,x;\alpha,\beta}) = \operatorname*{essinf}_{\tau\ge t} \operatorname*{esssup}_{\sigma\ge t} \exp\{Y_t^{(t,x);(\alpha,\sigma),(\beta,\tau)}\} = \operatorname*{esssup}_{\sigma\ge t} \operatorname*{essinf}_{\tau\ge t} \exp\{Y_t^{(t,x);(\alpha,\sigma),(\beta,\tau)}\}.$$

Next, for any $\alpha \in \mathcal{A}_t$ and $S_2 \in \mathbb{B}_t$, the formula (19) implies clearly that the first equality holds. The second one is treated in the same manner.

■

Remark 22 *The following relation holds: $\forall s \le T$, $\forall \tau, \sigma \in \mathcal{T}_t$,*

$$Y_s^{(t,x);(\alpha,\sigma),(\beta,\tau)} = \ln\Big(\mathbb{E}\Big[\exp\{h(X_\sigma^{t,x;\alpha,\beta})1_{[\sigma\le\tau<T]} + h'(X_\tau^{t,x;\alpha,\beta})1_{[\tau<\sigma]} + g(X_T^{t,x;\alpha,\beta})1_{[\sigma=\tau=T]} + \textstyle\int_s^{T\wedge\tau\wedge\sigma} \varphi(r, X_r^{t,x;\alpha,\beta}, \alpha_r, \beta_r)dr\}\Big|\, \mathcal{F}_s\Big]\Big).$$

Remark 23 *Of course, a more interesting and difficult issue is when the upper and the lower values of the mixed zero-sum two-players stochastic differential game are equal respectively to*

$$\inf_{(\beta,\tau)} \sup_{(\alpha,\sigma)} \Gamma(\alpha, \sigma; \beta, \tau) \quad \text{and} \quad \sup_{(\alpha,\sigma)} \inf_{(\beta,\tau)} \Gamma(\alpha, \sigma; \beta, \tau).$$

Unfortunately, this is still an open problem.

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